Nash Equilibrium in a Duopolistic Electricity Market

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Abstract

In this paper, we show a necessary and sufficient condition for the existence of Nash equilibrium in a non-cooperative game that describes a duopolistic electricity market. We also prove that the equilibrium is unique under the condition. Though the function we introduce for this purpose is defined implicitly, its inverse function has an explicit form as the sum of elementary functions. Basic properties of these functions are derived,and our main results are obtained with them.

1. Introduction

Recent studies of electricity markets can be classified into two major categories: those that investigate the structure of market power and those that analyze the properties of spot electricity price processes.

Research on market power is mainly based on oligopoly equilibrium models that explicitly describe the strategic behavior of energy suppliers. Some representative models are the supply function approach and the Cournot model. The supply function approach is an equilibrium pricing model where each firm can strategically offer its supply curve. This approach is based on Klemperer and Mayer [13] and has informed a series of papers that analyze electricity markets, the most well-known being Green and Newbery [4] and Newbery [16]. Green and Newbery [4] use a numerical model based on a supply function model to examine market power and the effect of entry in the British spot electricity market. Newbery [16] develops an extended supply function model that includes a spot market and contestable entry. On the other hand, Borenstein and Bushnell [2] present important factors that determine the impact of market power by using simulations based on a Cournot model.

Most empirical studies of the properties of spot electricity price movements, especially spikes, use stochastic processes (for example, Johnson and Barz [12], Barlow [1], Davison et al.[3], Huisman and Mahieu [7], Hadsell et al.[5], Hadsell and Shawky [6], Mount et al.[15], and Kanamura and Ohashi [14]).

Although the above two themes are connected, little attention has been paid to the relationship between market power and features of spot electricity price fluctuations. To examine this connection, Tezuka and Ishii [17] and Ishii [8] constructed another framework modeling the strategic behavior of various power producers. There are two notable advantages to this approach. First, it is comparatively easy to apply a wide variety of demand distributions, whereas the supply function approach is not tractable because of its complexity, and secondly, it covers many types of power generating firms. Nash equilibria within this framework are derived in Tezuka and Ishii [18], Ishii [9], Ishii and Tezuka [10], and Ishii and Tezuka [11].

In this study, we examine the existence of Nash equilibria in a non-cooperative game model, which is one aspect of this framework. The rest of this paper is organized as follows. In Section 2, we introduce an asymmetric duopolistic electricity market game model. In Section 3, we provide a necessary and sufficient condition for the existence of a Nash equilibrium in the game, and show that the equilibrium is unique. Section 4

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summarizes our findings and concludes the paper.

2. Model

In this section, we describe our assumptions and the notation used in this paper. We assume a duopolistic spot electricity market, so there are two power generating firms and many perfectly competitive retailers. Thus only two power generating firms supply electricity to the market, and all retailers buy from them. Each retailer has to buy electricity in the market to supply power to their customers.

For a one-period model, we call the beginning of the period *time zero* and the end of the period *time one*. At time zero, each power producer selects and offers a supply function strategically under uncertain demand, and the market maker adds individual supply functions to construct a market supply function. At time one, the quantity of demand will be realized, and the market maker will decide on a spot price with the market supply function and the realized demand. We note that it is easy to extend our approach to a multi-period model by the method described in Ishii [8].

Let $(\Omega, \mathcal{F}, P)$ be a probability space. As the total electricity demand, which is realized only at time one, is not foreseeable, it is modeled as a non-negative continuous random variable Y. On the other hand, all market participants know the probability distribution of Y at time zero.

Suppose that a_1 and a_2 are positive constants, and $b_1 \leq b_2$. For $j = 1, 2$, we set

$$f_j(x) = e^{a_j x + b_j} \qquad \text{for } x \in [0, \infty),$$

and

$$C_j(x) = \int_0^x f_j(u)du = \frac{1}{a_j}\left(e^{a_j x + b_j} - e^{b_j}\right) \qquad \text{for } x \in [0, \infty), \tag{1}$$

where f_j is the marginal cost function of power generating firm j and C_j is the cost function. Without loss of generality, we assume that the fixed cost is 0 for every firm. It is said that a typical generator's supply curve looks like a hockey stick, that is, the curve has a slight upward slope (or remains flat) until the generation capacity limit is reached, and then it jumps to infinity above the maximum output level. Since such functions make the model very complicated, it becomes difficult to derive equilibria. However, we can adjust a_j and b_j to produce various curves similar to hockey sticks, and then use exponential functions for the individual marginal cost functions.

We define $g_j : [0, \infty)^2 \to [0, \infty)$ by

$$g_j(x, \lambda_j) = e^{a_j x + b_j + \lambda_j} \qquad \text{for } (x, \lambda_j) \in [0, \infty)^2.$$

The economic interpretation of the function g_j is as follows. We view every $\lambda_j \in [0, \infty)$ as a strategy of power producer j. With the distribution of Y and information about the other power producer's strategy, power producer j selects a strategy λ_j, and bids his/her supply function $g_j(x, \lambda_j)$ at time zero. The expression $e^{\lambda_j} - 1$ can be interpreted as a rate of increase in the offered price.

In addition, $g_j(x, \lambda_j)$ has the inverse function for any $\lambda_j \in [0, \infty)$. We define $g_j^{-1}(z, \lambda_j)$ as follows:

$$g_j^{-1}(z, \lambda_j) = \begin{cases} 0 & \text{for } z \in [0, e^{b_j + \lambda_j}] \\ \dfrac{\log z - b_j - \lambda_j}{a_j} & \text{for } z \in (e^{b_j + \lambda_j}, \infty) \end{cases}. \tag{2}$$

We point out that the output is zero for each spot price $z \leq e^{b_j + \lambda_j}$.

Given any strategy vector $\lambda = (\lambda_1, \lambda_2) \in [0, \infty)^2$, we put

$$G(z, \lambda) := g_1^{-1}(z, \lambda_1) + g_2^{-1}(z, \lambda_2)$$

$$= \begin{cases} 0 & \text{for } z \in \left[0, \min\limits_{i=1,2}(e^{b_i+\lambda_i})\right] \\ \dfrac{\log z - b_{k(\lambda_1,\lambda_2)} - \lambda_{k(\lambda_1,\lambda_2)}}{a_{k(\lambda_1,\lambda_2)}} & \text{for } z \in \left(\min\limits_{i=1,2}(e^{b_i+\lambda_i}), \max\limits_{i=1,2}(e^{b_i+\lambda_i})\right], \\ \dfrac{(a_1+a_2)\log z - a_1(b_2+\lambda_2) - a_2(b_1+\lambda_1)}{a_1 a_2} & \text{for } z \in \left(\min\limits_{i=1,2}(e^{b_i+\lambda_i}), \infty\right) \end{cases} \tag{3}$$

where

$$k(\lambda_1, \lambda_2) = \begin{cases} 1 & \text{for } b_1 + \lambda_1 \le b_2 + \lambda_2 \\ 2 & \text{for } b_1 + \lambda_1 > b_2 + \lambda_2 \end{cases}.$$

We will refer to $G(z, \lambda)$ as the marketwide supply curve when the strategies of power producers 1 and 2 are λ_1 and λ_2, respectively.

Let $y>0$ be the realized electricity demand that all market participants observe at time one. The market maker solves the equation $G(z, \lambda)=y$ for z to set the quoted electricity spot price that balances the total supply with the total demand y. It is easy to show that a unique solution exists. Denoting the solution by $\varphi(y, \lambda)$, we have

$$\varphi(y, \lambda)$$

$$= \begin{cases} e^{a_{k(\lambda_1,\lambda_2)} y + b_{k(\lambda_1,\lambda_2)} + \lambda_{k(\lambda_1,\lambda_2)}} & \text{for } y \in \left(0, \dfrac{|b_1+\lambda_1-(b_2+\lambda_2)|}{a_{k(\lambda_1,\lambda_2)}}\right], \\ \exp\left\{\dfrac{a_1 a_2 y + a_1(b_2+\lambda_2) + a_2(b_1+\lambda_1)}{a_1+a_2}\right\} & \text{for } y \in \left(\dfrac{|b_1+\lambda_1-(b_2+\lambda_2)|}{a_{k(\lambda_1,\lambda_2)}}, \infty\right) \end{cases}. \tag{4}$$

Substituting (4) into (2) gives

$$g_1^{-1}(\varphi(y, \lambda), \lambda_1)$$

$$= \begin{cases} 0 & \text{for } y \in \left[0, \dfrac{b_1+\lambda_1-(b_2+\lambda_2)}{a_2}\right], \\ & \text{and } b_1+\lambda_1 > b_2+\lambda_2 \\ y & \text{for } y \in \left[0, \dfrac{b_2+\lambda_2-(b_1+\lambda_1)}{a_1}\right], \\ & \text{and } b_1+\lambda_1 \le b_2+\lambda_2 \\ \dfrac{a_2 y - (b_1+\lambda_1) + (b_2+\lambda_2)}{a_1+a_2} & \text{for } y \in \left(\dfrac{|b_1+\lambda_1-(b_2+\lambda_2)|}{a_{k(\lambda_1,\lambda_2)}}, \infty\right) \end{cases} \tag{5}$$

with $g_2^{-1}(\varphi(y, \lambda), \lambda_2)$ being expressed similarly. $g_j^{-1}(\varphi(y, \lambda), \lambda_j)$ denotes the supply of power producer j when the spot price at time one is $\varphi(y, \lambda)$.

From (1), (4) and (5), we define

$$F_1(y,\ \lambda) := \varphi(y,s) \cdot g_1^{-1}(\varphi(y,\lambda),\lambda_1) - C_1(g_1^{-1}(\varphi(y,\lambda),\lambda_1))$$

$$= \begin{cases} e^{a_1 y + b_1 + \lambda_1}\left(y - \dfrac{1}{a_1}e^{-\lambda_1}\right) + \dfrac{1}{a_1}e^{b_1} & \text{for } 0 < y \le \dfrac{b_2+\lambda_2-(b_1+\lambda_1)}{a_1} \text{ and } b_1+\lambda_1 < b_2+\lambda_2 \\ \exp\left\{\dfrac{a_1a_2\,y + a_1(b_2+\lambda_2) + a_2(b_1+\lambda_1)}{a_1+a_2}\right\}\left\{\dfrac{a_2\,y-(b_1+\lambda_1)+b_2+\lambda_2}{a_1+a_2} - \dfrac{1}{a_1}e^{-\lambda_1}\right\} + \dfrac{1}{a_1} & \\ \qquad\qquad \text{for } y > \dfrac{|b_1+\lambda_1-(b_2+\lambda_2)|}{a_{k(\lambda_1,\lambda_2)}}, & \\ 0 & \text{for } 0 < y \le \dfrac{b_1+\lambda_1-(b_2+\lambda_2)}{a_2} \text{ and } b_1+\lambda_1 \ge b_2+\lambda_2 \end{cases} \tag{6}$$

which represents the profit of producer 1 at time one. We can also define and calculate $F_2(y,\lambda)$ in the same manner.

3. Nash Equilibrium

In this section, we introduce a non-cooperative game that the two power generators are faced with. We then consider properties of a certain function that plays an important role in finding Nash equilibria in the game.

Let $\alpha \in (0,1)$, and suppose the following non-cooperative game. For each $j = 1, 2$,

$$\begin{cases} \text{the strategy set for player } j \text{ is } [0,\infty), \\ \text{the payoff to player } j \text{ is } \inf\{u \in \mathbf{R} \mid P(F_j(Y,\ \lambda) \le u) \ge \alpha\}. \end{cases} \tag{7}$$

In the above non-cooperative game, producers 1 and 2 are the players, and each strategically chooses a bid supply curve to maximize the α-quantile of his/her profit distribution given the probability distribution of total demand Y at time zero.

Lemma 1

Under the conditions given in Section 2, for $j = 1, 2$ and all $\lambda \in 0,\infty)^2$, we obtain

$$\inf\{u \in \mathbf{R} \mid P(F_j(Y,\ \lambda) \le u) \ge \alpha\} = F_j(y_\alpha,\ \lambda), \tag{8}$$

where y_α denotes the α-quantile of Y.

This lemma is verified in Ishii [8].

Now consider the following equation in one unknown x:

$$s_1 x + s_2 = s_3 e^{-x}, \tag{9}$$

where $s_1 > 0$, $s_2 \in \mathbf{R}$, and $s_3 > 0$.

Lemma 2

For $\forall s_1 > 0$, $\forall s_2 \in \mathbf{R}$, and $\forall s_3 > 0$, equation (9) has a unique solution.

PROOF. Let $h(x) = s_1 x + s_2 - s_3 e^{-x}$. Since

$$\frac{dh}{dx}(x) = s_1 + s_3 e^{-x} > 0 \quad \text{for } x \in \mathbf{R},$$

h is a strictly increasing function. In addition, $\lim_{x\to\infty} h(x) = \infty$ and $\lim_{x\to-\infty} h(x) = -\infty$. Hence there exists a unique solution satisfying $h(x) = 0$. We denote the solution by $\xi(s_1, s_2, s_3)$. □

The function ξ is used to derive Nash equilibria in the non-cooperative game (7), so we will derive some properties of ξ.

From equation (9), we have the following lemmas.

Lemma 3

$\xi(s_1, s_2, s_3) > 0$ if and only if $s_2 < s_3$.

Lemma 4

The function ξ is strictly decreasing with respect to s_2, and is strictly increasing with respect to s_3.

PROOF. By the implicit function theorem, the first partial derivatives of ξ are

$$\frac{\partial \xi}{\partial s_2}(s_1, s_2, s_3) = -\frac{1}{s_1 + s_3 e^{-\xi(s_1, s_2, s_3)}} < 0,$$
$$\frac{\partial \xi}{\partial s_3}(s_1, s_2, s_3) = \frac{e^{-\xi(s_1, s_2, s_3)}}{s_1 + s_3 e^{-\xi(s_1, s_2, s_3)}} > 0,$$

from which the desired results are obtained. □

We now put ξ_1 and ξ_2 as follows:

$$\xi_1(u) = \xi\left(\frac{a_1}{a_1+a_2}, -\frac{a_1(a_1 y + b_1 - b_2 + u)}{a_1+a_2} + 1, 1\right) \quad \text{for } u \in \mathbf{R},$$
$$\xi_2(u) = \xi\left(\frac{a_2}{a_1+a_2}, -\frac{a_2(a_2 y - b_1 + b_2 + u)}{a_1+a_2} + 1, 1\right) \quad \text{for } u \in \mathbf{R}.$$

Some remarks about ξ_j are necessary here to avoid misunderstanding and confusion. Strictly speaking, ξ_j depends on not only u but also a_1, a_2, b_1, b_2 and y. For convenience, the expression $\xi_j(u)$ is used in this paper. In the two theorems as stated below, a value y_α is assigned to the variable y in ξ_j.

Lemma 5

For each $j = 1, 2$, ξ_j is a strictly increasing convex function, and the first derivative is less than 1.

PROOF. Using Lemma 4, the definition of ξ_1 and the chain rule, we have that

$$\frac{d\xi_1}{du}(u) = \frac{\partial \xi}{\partial s_2}\left(\frac{a_1}{a_1+a_2}, -\frac{a_1(a_1 y + b_1 - b_2 + u)}{a_1+a_2} + 1, 1\right) \cdot \frac{-1}{a_1+a_2}$$
$$= \frac{1}{1 + \frac{a_1+a_2}{a_1} e^{-\xi_1(u)}} \quad .$$

It is clear that $0 < \frac{d\xi_1}{du}(u) < 1$, and so ξ_1 is strictly increasing. Since the second derivative of ξ_1 is

$$\frac{d^2\xi_1}{du^2}(u) = \frac{\frac{a_1+a_2}{a_1} e^{-\xi_1(u)}}{\left\{1 + \frac{a_1+a_2}{a_1} e^{-\xi_1(u)}\right\}^3} > 0,$$

then ξ_1 is convex. For ξ_2, we can show the result similarly. □

Lemma 6

The inverses of ξ_1 and ξ_2 are

$$\xi_1^{-1}(v) = v-(a_1 y+b_1-b_2)+\frac{a_1+a_2}{a_1}(1-e^{-v}) \quad \text{for } v \in \mathbf{R},$$

$$\xi_2^{-1}(v) = v-(a_2 y-b_1+b_2)+\frac{a_1+a_2}{a_2}(1-e^{-v}) \quad \text{for } v \in \mathbf{R}.$$

PROOF. It is apparent that the image set of ξ_1 is $\mathbf{R}$. So by Lemma 5, the inverse of ξ_1 exists and its domain is $\mathbf{R}$. We also have

$$\begin{aligned}\xi_1^{-1}(\xi_1(u)) &= \xi_1(u)-(a_1 y+b_1-b_2)+\frac{a_1+a_2}{a_1}(1-e^{-\xi_1(u)})\\ &=\frac{a_1+a_2}{a_1}\left\{\frac{a_1}{a_1+a_2}\xi_1(u)-\frac{a_1(a_1 y+b_1-b_2+u)}{a_1+a_2}+1-e^{-\xi_1(u)}\right\}+u\\ &=u.\end{aligned}$$

The proof for ξ_2 is straightforward. □

We also note that $\xi_2^{-1}(0) = -(a_2 y+b_1-b_2) < 0$, so that $\xi_2(0) > 0$.

Lemma 7

Fix $y > 0$ and $\lambda_2 \geq 0$. If $-a_1 y-b_1+b_2+\lambda_2 \geq \xi_2(\lambda_2)$, then $\lambda_1 = -a_1 y-b_1+b_2+\lambda_2$ is the unique maximum point for $F_1(y, \lambda)$. Otherwise, $F_1(y, \lambda)$ is maximized uniquely at $\lambda_1 = \xi_2(\lambda_2)$.

PROOF. By (6) and $-a_1 y-b_1+b_2+\lambda_2 < -b_1+b_2+\lambda_2 < a_2 y-b_1+b_2+\lambda_2$,

$$F_1(y,\lambda)=\begin{cases} ye^{a_1 y+b_1+\lambda_1}-\dfrac{e^{b_1}}{a_1}(e^{a_1 y}-1) & \text{for } 0 \leq \lambda_1 \leq -a_1 y-b_1+b_2+\lambda_2\\ \exp\left\{\dfrac{a_1 a_2 y+a_1(b_2+\lambda_2)+a_2(b_1+\lambda_1)}{a_1+a_2}\right\}\left\{\dfrac{a_2 y-(b_1+\lambda_1)+b_2+\lambda_2}{a_1+a_2}-\dfrac{1}{a_1}e^{-\lambda_1}\right\}+\dfrac{1}{a_1}e^{b_1} & \text{for } -a_1 y-b_1+b_2+\lambda_2 < \lambda_1 \leq a_2 y-b_1+b_2+\lambda_2\\ 0 & \text{for } a_2 y-b_1+b_2+\lambda_2 < \lambda_1 \end{cases} \tag{10}$$

For $\lambda_1 \in (-a_1 y-b_1+b_2+\lambda_2, a_2 y-b_1+b_2+\lambda_2]$,

$$\begin{aligned}\frac{\partial}{\partial\lambda_1}F_1(y,\lambda) &= \frac{1}{a_1+a_2}\exp\left\{\frac{a_1 a_2 y+a_1(b_2+\lambda_2)+a_2(b_1+\lambda_1)}{a_1+a_2}\right\}\\ &\quad\times\left\{-\frac{a_2}{a_1+a_2}\lambda_1+\frac{a_2(a_2 y-b_1+b_2+\lambda_2)}{a_1+a_2}-1+e^{-\lambda_1}\right\}.\end{aligned}$$

Here,

$$-\frac{a_2}{a_1+a_2}\lambda_1+\frac{a_2(a_2 y-b_1+b_2+\lambda_2)}{a_1+a_2}-1+e^{-\lambda_1}\begin{cases}>0 & \text{for } \lambda_1 < \xi_2(\lambda_2)\\ =0 & \text{for } \lambda_1 = \xi_2(\lambda_2).\\ <0 & \text{for } \lambda_1 > \xi_2(\lambda_2)\end{cases}$$

Then $\lambda_1 = \xi_2(\lambda_2)$ is the unique maximizer when $-a_1 y-b_1+b_2+\lambda_2 < \xi_2(\lambda_2) \leq a_2 y-b_1+b_2+\lambda_2$, so we have to check that this holds. As it is apparent that $a_2 y-b_1+b_2+\lambda_2 > 0$, Lemma 3 gives us

$$a_2\, y - b_1 + b_2 + \lambda_2 > 0$$
$$\Leftrightarrow -\frac{a_2\,(a_2\, y - b_1 + b_2 + \lambda_2)}{a_1 + a_2} + 1 < 1$$
$$\Leftrightarrow \xi_2(\lambda_2) > 0.$$

By substituting $\lambda_1 = a_2 y - b_1 + b_2 + \lambda_2$ for the above partial derivative,

$$\frac{\partial}{\partial \lambda_1} F_1(y, \lambda)\Big|_{\lambda_1 = a_2 y - b_1 + b_2 + \lambda_2} = \frac{1}{a_1 + a_2}\, e^{a_2 y + b_2 + \lambda_2}\left\{-1 + e^{-a_2 y + b_1 - b_2 - \lambda_2}\right\} < 0\,.$$

From the above inequalities,

$$0 < \xi_2(\lambda_2) < a_2 y - b_1 + b_2 + \lambda_2.$$

Therefore, $\lambda_1 = -a_1 y - b_1 + b_2 + \lambda_2$ is the unique maximum point for $F_1(y, \lambda)$ when $-a_1 y - b_1 + b_2 + \lambda_2 \geq \xi_2(\lambda_2)$, and otherwise $F_1(y, \lambda)$ is maximized uniquely at $\lambda_1 = \xi_2(\lambda_2)$. □

Lemma 8

Fix $y > 0$ and $\lambda_1 \geq 0$.

(i) If $a_1 y + b_1 + \lambda_1 - b_2 \leq 0$, $F_2(y, \lambda) = 0$ for any $\lambda_2 \in [0, \infty)$.

(ii) If $a_1 y + b_1 + \lambda_1 - b_2 > 0$, the unique maximum point for $F_2(y, \lambda)$ is

$$\begin{cases} \lambda_2 = -a_2 y + b_1 + \lambda_1 - b_2 & \text{for } -a_2 y + b_1 + \lambda_1 - b_2 \geq \xi_1(\lambda_1), \\ \lambda_2 = \xi_1(\lambda_1) & \text{for } -a_2 y + b_1 + \lambda_1 - b_2 < \xi_1(\lambda_1). \end{cases}$$

Proof. By (6) and $-a_2 y + b_1 + \lambda_1 - b_2 < b_1 + \lambda_1 - b_2 < a_1 y + b_1 + \lambda_1 - b_2$,

$$F_2(y, \lambda) = \begin{cases} y e^{a_2 y + b_2 + \lambda_2} - \dfrac{e^{b_2}}{a_2}\,(e^{a_2 y} - 1) & \text{for } 0 \leq \lambda_2 \leq -a_2 y - b_1 + \lambda_1 - b_2 \\ \exp\left\{\dfrac{a_1 a_2 y + a_1(b_2 + \lambda_2) + a_2(b_1 + \lambda_1)}{a_1 + a_2}\right\}\left\{\dfrac{a_1 y - (b_2 + \lambda_2) + b_1 + \lambda_1}{a_1 + a_2} - \dfrac{1}{a_2} e^{-\lambda_2}\right\} + \dfrac{1}{a_2} e^{b_2} & \text{for } -a_2 y + b_1 + \lambda_1 - b_2 < \lambda_2 \leq a_1 y + b_1 + \lambda_1 - b_2 \\ 0 & \text{for } a_1 y + b_1 + \lambda_1 - b_2 < \lambda_2 \end{cases} \tag{11}$$

Then $a_1 y + b_1 + \lambda_1 - b_2 \leq 0$ implies $F_2(y, \lambda) = 0$ for any $\lambda_2 \in [0, \infty)$. As in the proof of Lemma 7, we have

$$a_1 y + b_1 + \lambda_1 - b_2 > 0 \Leftrightarrow \xi_1(\lambda_1) > 0,$$

from which we can show (ii). □

Lemma 9

(i) If $0 < a_2 y < 1$,

$$-a_1 y - b_1 + b_2 + \lambda_2 \geq \xi_2(\lambda_2) \Leftrightarrow \lambda_2 \geq -\log(-a_2 y + 1) + a_1 y + b_1 - b_2.$$

(ii) If $a_2 y \geq 1$,

$$-a_1 y - b_1 + b_2 + \lambda_2 < \xi_2(\lambda_2) \quad \text{for any } \lambda_2 \in [0, \infty).$$

(iii) If $0 < a_1 y < 1$,

$$-a_2 y + b_1 + \lambda_1 - b_2 \geq \xi_1(\lambda_1) \Leftrightarrow \lambda_1 \geq -\log(-a_1 y + 1) + a_2 y - b_1 + b_2.$$

(iv) If $a_1 y \geq 1$,

$$-a_2 y + b_1 + \lambda_1 - b_2 < \xi_1(\lambda_1) \quad \text{for any } \lambda_1 \in [0, \infty).$$

PROOF. Under the assumption that $0 < a_2 y < 1$, we have by the definition of ξ_2 that

$$\begin{aligned}
&-a_1 y - b_1 + b_2 + \lambda_2 \geq \xi_2(\lambda_2)\\
\Leftrightarrow\ &\frac{a_2(-a_1 y - b_1 + b_2 + \lambda_2)}{a_1 + a_2} - \frac{a_2(a_2 y - b_1 + b_2 + \lambda_2)}{a_1 + a_2} + 1 \geq e^{-(-a_1 y - b_1 + b_2 + \lambda_2)}\\
\Leftrightarrow\ &a_2 y + 1 \geq e^{a_1 y + b_1 - b_2 - \lambda_2}\\
\Leftrightarrow\ &\lambda_2 \geq -\log(-a_2 y + 1) + a_1 y + b_1 - b_2.
\end{aligned}$$

Thus (i) is obtained, and (iii) can be proved similarly.

Now suppose $a_2 y \geq 1$. For each $\lambda_2 \in [0, \infty)$,

$$-a_2 y + 1 \leq 0 < e^{a_1 y + b_1 - b_2 - \lambda_2}.$$

Therefore

$$-a_1 y - b_1 + b_2 + \lambda_2 < \xi_2(\lambda_2),$$

and so (ii) is obtained. The proof of (iv) is similar. □

We now let S_j be the best response set of player j for each $j = 1, 2$.

Theorem 1

The best response set of each player is derived as follows.

(i) If $0 < a_1 y_\alpha < 1, 0 < a_2 y_\alpha < 1$, and $-\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 \leq 0$, then

$$\begin{aligned}
S_1 &= \{(-a_1 y_\alpha - b_1 + b_2 + \lambda_2, \lambda_2) \mid \lambda_2 \geq 0\},\\
S_2 &= \{(\lambda_1, \lambda_2) \mid 0 \leq \lambda_1 \leq -a_1 y_\alpha - b_1 + b_2, \lambda_2 \geq 0\}\\
&\quad \cup \{(\lambda_1, \xi_1(\lambda_1)) \mid -a_1 y_\alpha - b_1 + b_2 < \lambda_1 < -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2\}\\
&\quad \cup \{(\lambda_1, -a_2 y_\alpha + b_1 + \lambda_1 - b_2) \mid -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2 \leq \lambda_1\}.
\end{aligned}$$

(ii) If $0 < a_1 y_\alpha < 1, 0 < a_2 y_\alpha < 1$ and $-\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 > 0$, then

$$\begin{aligned}
S_1 &= \{(\xi_2(\lambda_2), \lambda_2) \mid 0 \leq \lambda_2 < -\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2\}\\
&\quad \cup \{(-a_1 y_\alpha - b_1 + b_2 + \lambda_2, \lambda_2) \mid -\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 \leq \lambda_2\}\\
S_2 &= \{(\lambda_1, \lambda_2) \mid 0 \leq \lambda_1 \leq -a_1 y_\alpha - b_1 + b_2, \lambda_2 \geq 0\}\\
&\quad \cup \{(\lambda_1, \xi_1(\lambda_1)) \mid -a_1 y_\alpha - b_1 + b_2 < \lambda_1 < -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2, \lambda_1 \geq 0\}\\
&\quad \cup \{(\lambda_1, -a_2 y_\alpha + b_1 + \lambda_1 - b_2) \mid -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2 \leq \lambda_1\}.
\end{aligned}$$

(iii) If $a_1 y_\alpha \geq 1, 0 < a_2 y_\alpha < 1$ and $-\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 \leq 0$, then

$$\begin{aligned}
S_1 &= \{(-a_1 y_\alpha - b_1 + b_2 + \lambda_2, \lambda_2) \mid \lambda_2 \geq 0\},\\
S_2 &= \{(\lambda_1, \lambda_2) \mid 0 \leq \lambda_1 \leq -a_1 y_\alpha - b_1 + b_2, \lambda_2 \geq 0\} \cup \{(\lambda_1, \xi_1(\lambda_1)) \mid -a_1 y_\alpha - b_1 + b_2 < \lambda_1\}.
\end{aligned}$$

(iv) If $a_1 y_\alpha \geq 1, 0 < a_2 y_\alpha < 1$ and $-\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 > 0$, then

$$\begin{aligned}
S_1 &= \{(\xi_2(\lambda_2), \lambda_2) \mid 0 \leq \lambda_2 < -\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2\}\\
&\quad \cup \{(-a_1 y_\alpha - b_1 + b_2 + \lambda_2, \lambda_2) \mid -\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 \leq \lambda_2\},\\
S_2 &= \{(\lambda_1, \lambda_2) \mid 0 \leq \lambda_1 \leq -a_1 y_\alpha - b_1 + b_2, \lambda_2 \geq 0\} \cup \{(\lambda_1, \xi_1(\lambda_1)) \mid -a_1 y_\alpha - b_1 + b_2 < \lambda_1, \lambda_1 \geq 0\}.
\end{aligned}$$

(v) If $0 < a_1 y_\alpha < 1$ and $a_2 y_\alpha \geq 1$, then

$$\begin{aligned}
S_1 &= \{(\xi_2(\lambda_2), \lambda_2) \mid 0 \leq \lambda_2\},\\
S_2 &= \{(\lambda_1, \lambda_2) \mid 0 \leq \lambda_1 \leq -a_1 y_\alpha - b_1 + b_2, \lambda_2 \geq 0\}\\
&\quad \cup \{(\lambda_1, \xi_1(\lambda_1)) \mid -a_1 y_\alpha - b_1 + b_2 < \lambda_1 < -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2, \lambda_1 \geq 0\}\\
&\quad \cup \{(\lambda_1, -a_2 y_\alpha + b_1 + \lambda_1 - b_2) \mid -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2 \leq \lambda_1\}.
\end{aligned}$$

(vi) If $a_1 y_\alpha \geq 1$ and $a_2 y_\alpha \geq 1$, then

$$S_1 = \{(\xi_2(\lambda_2),\ \lambda_2) \mid 0 \leq \lambda_2\},$$

$$S_2 = \{(\lambda_1,\ \lambda_2) \mid 0 \leq \lambda_1 \leq -a_1 y_\alpha - b_1 + b_2,\ \lambda_2 \geq 0\} \cup \{(\lambda_1,\ \xi_1(\lambda_1)) \mid -a_1 y_\alpha - b_1 + b_2 < \lambda_1,\ \lambda_1 \geq 0\}.$$

PROOF. (i) : From the assumption that $-\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 \leq 0$, we have

$$-\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 \leq \lambda_2.$$

By this inequality and $0 < a_2 y_\alpha < 1$, Lemma 9 (i) gives us

$$-a_1 y_\alpha - b_1 + b_2 + \lambda_2 \geq \xi_2(\lambda_2).$$

Lemma 7 implies that $F_1(y_\alpha, \lambda)$ has a unique maximum at $\lambda_1 = -a_1 y_\alpha - b_1 + b_2 + \lambda_2$, and so S_1 is represented above.

It is apparent that $0 < -\log(-a_2 y_\alpha + 1) \leq -a_1 y_\alpha - b_1 + b_2$. Then, if $0 \leq \lambda_1 \leq -a_1 y - b_1 + b_2$, Lemma 8 (i) gives us

$$F_2(y_\alpha, \lambda) = 0 \quad \text{for } \lambda_2 \in [0, \infty).$$

By Lemma 9 (iii), $-a_1 y_\alpha - b_1 + b_2 < \lambda_1 < -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2$ leads to $-a_2 y_\alpha + b_1 + \lambda_1 - b_2 < \xi_1(\lambda_1)$. Then, by Lemma 8 (ii), $F_2(y_\alpha, \lambda)$ is uniquely maximized at $\lambda_2 = \xi_1(\lambda_1)$. Similarly, $F_2(y_\alpha, \lambda)$ is uniquely maximized at $\lambda_2 = -a_2 y_\alpha + b_1 + \lambda_1 - b_2$ for $\lambda_1 \geq -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2$. Thus S_2 is derived as above.

(ii) : For any $\lambda_2 \in [0, -\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2)$,

$$-a_1 y_\alpha - b_1 + b_2 + \lambda_2 < \xi_2(\lambda_2)$$

from $0 < a_2 y_\alpha < 1$ and Lemma 9 (i). By Lemma 7, $F_1(y_\alpha, \lambda)$ is uniquely maximized at $\lambda_1 = \xi_2(\lambda_2)$. For any $\lambda_2 \geq -\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2$, we get similarly that $F_1(y_\alpha, \lambda)$ is uniquely maximized at $\lambda_1 = -a_1 y_\alpha - b_1 + b_2 + \lambda_2$. Hence S_1 is obtained, and S_2 is derived similarly to (i).

(iii) : S_1 is derived as in (i). From $a_1 y_\alpha \geq 1$ and Lemma 9 (iv), we have that

$$-a_2 y_\alpha + b_1 + \lambda_1 - b_2 < \xi_1(\lambda_1) \quad \text{for } \lambda_1 \geq 0.$$

Then $\lambda_2 = \xi_1(\lambda_1)$ is the unique maximum point for $F_2(y_\alpha, \lambda)$, and S_2 is obtained.

(iv) : In this case, we can derive S_1 and S_2 as in (ii) and (iii) respectively.

(v) : From $a_2 y_\alpha \geq 1$ and Lemma 9 (ii),

$$-a_1 y_\alpha - b_1 + b_2 + \lambda_2 < \xi_2(\lambda_2) \quad \text{for } \lambda_2 \geq 0.$$

Recall that $F_1(y_\alpha, \lambda)$ has a unique maximum at $\lambda_1 = \xi_2(\lambda_2)$ by Lemma 7. Thus we have S_1. The proof for S_2 is handled as in (i).

(vi) : S_1 and S_2 are verified in a similar way to (v) and (iii), respectively. □

Lemma 10

If $0 < a_1 y < 1$, we have following two inequalities:

$$\xi_2(0) < -\log(-a_1 y + 1) + a_2 y - b_1 + b_2, \tag{12}$$

$$\xi_2(-\log(-a_1 y + 1)) < -\log(-a_1 y + 1) + a_2 y - b_1 + b_2. \tag{13}$$

PROOF. Inequality (12) is shown as follows.

$$\frac{a_2}{a_1 + a_2}\{-\log(-a_1 y + 1) + a_2 y - b_1 + b_2\} - \frac{a_2(a_2 y - b_1 + b_2)}{a_1 + a_2} + 1$$

$$= \frac{-a_2 \log(-a_1 y + 1)}{a_1 + a_2} + 1 > 1 > e^{-\{-\log(-a_1 y + 1) + a_2 y - b_1 + b_2\}}$$

So from the definition of ξ_2,

$$\xi_2(0) = \xi\left(\frac{a_2}{a_1+a_2},\ -\frac{a_2(a_2\,y-b_1+b_2)}{a_1+a_2}+1,\ 1\right) < -\log(-a_1y+1)+a_2y-b_1+b_2\,.$$

Inequality (13) is derived similarly. □

Lemma 11

If $0 < a_2y < 1$ and $-\log(-a_2y+1)+a_1y+b_1-b_2 > 0$,

$$\xi_1(-\log(-a_2y+1)) < -\log(-a_2y+1)+a_1y+b_1-b_2. \tag{14}$$

The proof is essentially the same as the proof of Lemma 10.

Lemma 12

Consider the following system of equations in two unknowns v_1 and v_2:

$$\begin{cases} v_1 = \xi_2(v_2) \\ v_2 = \xi_1(v_1) \end{cases}. \tag{15}$$

(i) If $y \geq \dfrac{1}{a_1}+\dfrac{1}{a_2}$, then (15) is inconsistent.

(ii) If $y < \dfrac{1}{a_1}+\dfrac{1}{a_2}$ and $1-a_2y > e^{a_1y+b_1-b_2}$, then (15) cannot have a solution in $[0,\infty)^2$.

(iii) If $y < \dfrac{1}{a_1}+\dfrac{1}{a_2}$ and $1-a_2y \leq e^{a_1y+b_1-b_2}$, then (15) has a unique solution in $[0,\infty)^2$.

Proof. It is apparent that the original system has exactly the same solutions as

$$\begin{cases} v_1 = \xi_1^{-1}(v_2) = v_2-(a_1y+b_1-b_2)+\dfrac{a_1+a_2}{a_1}(1-e^{-v_2}) \\ v_2 = \xi_2^{-1}(v_1) = v_1-(a_2y-b_1+b_2)+\dfrac{a_1+a_2}{a_2}(1-e^{-v_1}) \end{cases}. \tag{16}$$

This enables us to show the desired results by (16).

Define $l_1, l_2 : \mathbf{R} \to \mathbf{R}$ as follows:

$$l_1(u) = u+a_1y+b_1-b_2-\frac{a_1+a_2}{a_1},$$

$$l_2(u) = u+a_2y-b_1+b_2-\frac{a_1+a_2}{a_2}.$$

The inverses of these functions are

$$l_1^{-1}(v) = v-(a_1y+b_1-b_2)+\frac{a_1+a_2}{a_1},$$

$$l_2^{-1}(v) = v-(a_2y-b_1+b_2)+\frac{a_1+a_2}{a_2}.$$

By Lemma 6, l_j^{-1} is an asymptote of ξ_j^{-1} for $j = 1, 2$. Since

$$l_j^{-1}(v) > \xi_j^{-1}(v) \quad \text{for } v \in \mathbf{R},$$

we have

$$l_j(u) < \xi_j(u) \quad \text{for } u \in \mathbf{R}.$$

(i) : Rearranging $y \geq \frac{1}{a_1} + \frac{1}{a_2}$, we get.

$$-(a_2 y - b_1 + b_2) + \frac{a_1+a_2}{a_2} \leq a_1 y + b_1 - b_2 - \frac{a_1+a_2}{a_1}.$$

This implies

$$\begin{aligned} \xi_2^{-1}(v) < l_2^{-1}(v) &= v - (a_2 y - b_1 + b_2) + \frac{a_1+a_2}{a_2} \\ &\leq v + a_1 y + b_1 - b_2 - \frac{a_1+a_2}{a_1} \\ &= l_1(v) < \xi_1(v). \end{aligned}$$

Thus we conclude that the system (16) has no solution.

(ii) : As in (i), we have from $y < \frac{1}{a_1} + \frac{1}{a_2}$ that

$$l_2^{-1}(u) > l_1(u) \quad \text{for } u \in \mathbf{R}.$$

The inequality $1 - a_2 y > e^{a_1 y + b_1 - b_2}$ implies that

$$0 < a_2 y < 1 \text{ and } a_1 y + b_1 - b_2 < 0. \tag{17}$$

In addition

$$1 - a_2 y > e^{a_1 y + b_1 - b_2}$$

$$-(a_1 y + b_1 - b_2) - (a_2 y - b_1 + b_2) + \frac{a_1+a_2}{a_2}(1 - e^{a_1 y + b_1 - b_2}) > 0$$

$$\xi_2^{-1}(\xi_1^{-1}(0)) > 0. \tag{18}$$

On the other hand, from (17)

$$\xi_1^{-1}(0) = -(a_1 y + b_1 - b_2) > 0,$$

so that

$$\{u \geq 0 \mid \xi_1(u) \geq 0\} = \{u \mid u \geq -(a_1 y + b_1 - b_2)\}. \tag{19}$$

We have by Lemma 5 that

$$\frac{d\xi_1}{du}(u) < 1 \quad \text{for } u \in \mathbf{R}, \tag{20}$$

and by Lemma 6 that

$$\frac{d}{du}\xi_2^{-1}(u) = 1 + \frac{a_1+a_2}{a_2} e^{-u} > 1, \quad \text{for } u \in \mathbf{R}. \tag{21}$$

So combining (18), (19), (20) and (21), we get

$$\xi_2^{-1}(u) > \xi_1(u) \quad \text{for } u \geq -(a_1 y + b_1 - b_2).$$

Thus there is no solution in $[0,\infty)^2$.

(iii) : As stated in the proof of (ii),

$$l_2^{-1}(u) > l_1(u) \quad \text{for } \forall u \in \mathbf{R}.$$

So there exists $u_0 \in \mathbf{R}$ such that

$$u > u_0 \Rightarrow \xi_2^{-1}(u) > \xi_1(u). \tag{22}$$

Recall that $\xi_1(0)$ is the unique solution to the equation

$$\frac{a_1}{a_1+a_2}x-\frac{a_1(a_1y+b_1-b_2)}{a_1+a_2}+1=e^{-x}.$$

By substituting $\xi_2^{-1}(0)=-(a_2y-b_1+b_2)$ on the left hand side, we get

$$\frac{a_1}{a_1+a_2}\xi_2^{-1}(0)-\frac{a_1(a_1y+b_1-b_2)}{a_1+a_2}+1=-a_1y+1<e^{a_2y-b_1+b_2}=e^{-\xi_2^{-1}(0)},$$

that is,

$$\xi_2^{-1}(0)<\xi_1(0). \tag{23}$$

Because of (22) and (23), we conclude that the system (16) has a solution in $[0,\infty)^2$. Furthermore, (20) and (21) imply that this solution must be unique. □

Theorem 2

The non-cooperative game (7) has Nash equilibrium if and only if $y_\alpha<\frac{1}{a_1}+\frac{1}{a_2}$. Furthermore, when the equilibrium exists, it is unique and non-negative.

PROOF. For each case in Theorem 1, we examine the existence of the Nash equilibrium.

(i): Suppose that $0<a_1y_\alpha<1$, $0<a_2y_\alpha<1$, and $-\log(-a_2y_\alpha+1)+a_1y_\alpha+b_1-b_2\le 0$. It is apparent that $0<-\log(-a_2y_\alpha+1)\le -a_1y_\alpha-b_1+b_2$ and $\xi_1(-a_1y_\alpha-b_1+b_2)=0$. Since the first derivative of ξ_1 is less than 1 from Lemma 5, we have that

$$S_1\cap S_2=\{(-a_1y_\alpha-b_1+b_2,0)\}.$$

(ii): Next we assume that $0<a_1y_\alpha<1$, $0<a_2y_\alpha<1$ and $-\log(-a_2y_\alpha+1)+a_1y_\alpha+b_1-b_2>0$. From the third inequality and Lemma 9 (i),

$$-a_1y_\alpha-b_1+b_2<\xi_2(0), \tag{24}$$

so that

$$0=\xi_1(-a_1y_\alpha-b_1+b_2)<\xi_1(\xi_2(0)).$$

Then

$$\xi_2^{-1}(\xi_2(0))=0<\xi_1(\xi_2(0)). \tag{25}$$

Lemma 10 gives us that

$$\xi_2(0)<-\log(-a_1y_\alpha+1)+a_2y_\alpha-b_1+b_2 \tag{26}$$

and

$$\begin{aligned}&\xi_1(-\log(-a_1y_\alpha+1)+a_2y_\alpha-b_1+b_2)\\&=-\log(-a_1y_\alpha+1)\\&=\xi_2^{-1}(\xi_2(-\log(-a_1y_\alpha+1)))\\&<\xi_2^{-1}(-\log(-a_1y_\alpha+1)+a_2y_\alpha-b_1+b_2).\end{aligned} \tag{27}$$

Because of Lemma 11,

$$\xi_1(-\log(-a_2y_\alpha+1))<-\log(-a_2y_\alpha+1)+a_1y_\alpha+b_1-b_2=\xi_2^{-1}(-\log(-a_2y_\alpha+1)). \tag{28}$$

Applying (24), (25), (26), (27) and (28), we deduce that there exists a unique $\lambda_1^*\in(\xi_2(0),\min(-\log(-a_2y_\alpha+1),-\log(-a_1y_\alpha+1)+a_2y_\alpha-b_1+b_2))$, which satisfies $\xi_1(\lambda_1^*)=\xi_2^{-1}(\lambda_1^*)$. Putting $\lambda_2^*=\xi_1(\lambda_1^*)$, we conclude that

$$S_1\cap S_2=\{(\lambda_1^*,\lambda_2^*)\}.$$

(iii): From the assumptions that $a_1 y_\alpha \geq 1$, $0 < a_2 y_\alpha < 1$ and $-\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 \leq 0$, we can prove that

$$S_1 \cap S_2 = \{(-a_1 y_\alpha - b_1 + b_2, 0)\}$$

as in (i).

(iv): We suppose that $a_1 y_\alpha \geq 1$, $0 < a_2 y_\alpha < 1$ and $-\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 > 0$.

Combining the third inequality and Lemma 9 (i) yields

$$-a_1 y_\alpha - b_1 + b_2 < \xi_2(0). \tag{29}$$

Since $\xi_2(0) > 0$ and ξ_1 is strictly increasing,

$$\xi_2^{-1}(\xi_2(0)) = 0 = \xi_1(-a_1 y_\alpha - b_1 + b_2) < \xi_1(\xi_2(0)). \tag{30}$$

Lemma 11 gives us that

$$\xi_1(-\log(-a_2 y_\alpha + 1)) < -\log(-a_2 y_\alpha + 1) + a_1 y_\alpha + b_1 - b_2 = \xi_2^{-1}(-\log(-a_2 y_\alpha + 1)). \tag{31}$$

Therefore (29), (30) and (31) together imply that there exists a unique $\lambda_1^* \in (\xi_2(0), -\log(-a_2 y_\alpha + 1))$ satisfying $\xi_1(\lambda_1^*) = \xi_2^{-1}(\lambda_1^*)$. Hence, we obtain

$$S_1 \cap S_2 = \{(\lambda_1^*, \lambda_2^*)\},$$

where $\lambda_2^* = \xi_1(\lambda_1^*)$.

(v): Suppose that $0 < a_1 y_\alpha < 1$ and $a_2 y_\alpha \geq 1$. Applying Lemma 9 (ii) and Lemma 10, we get

$$-a_1 y_\alpha - b_1 + b_2 < \xi_2(0) < -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2. \tag{32}$$

Then, by the monotonicity of ξ_1,

$$\xi_2^{-1}(\xi_2(0)) = 0 = \xi_1(-a_1 y_\alpha - b_1 + b_2) < \xi_1(\xi_2(0)). \tag{33}$$

In addition, $-\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2 > 0$ implies that

$$\begin{aligned} &\xi_1(-\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2) \\ &= -\log(-a_1 y_\alpha + 1) \\ &< -\log(-a_1 y_\alpha + 1) + \frac{a_1 + a_2}{a_1}(1 - e^{-\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2}) \\ &= \xi_2(-\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2) \end{aligned} \tag{34}$$

Combining (32), (33) and (34), we conclude that there exists a unique $\lambda_1^* \in (\xi_2(0), -\log(-a_1 y_\alpha + 1) + a_2 y_\alpha - b_1 + b_2)$ satisfying $\xi_1(\lambda_1^*) = \xi_2^{-1}(\lambda_1^*)$. Therefore,

$$S_1 \cap S_2 = \{(\lambda_1^*, \lambda_2^*)\},$$

where $\lambda_2^* = \xi_1(\lambda_1^*)$.

(vi): Finally, we consider the case where both $a_1 y_\alpha \geq 1$ and $a_2 y_\alpha \geq 1$ hold. By Lemma 9 (ii),

$$-a_1 y_\alpha - b_1 + b_2 < \xi_2(0).$$

It is obvious that $1 - a_2 y_\alpha \leq e^{a_1 y_\alpha + b_1 - b_2}$. Therefore, Lemma 12 tells us that

$$y_\alpha \geq \frac{1}{a_1} + \frac{1}{a_2} \quad \Rightarrow \quad \text{the game (7) has no Nash equilibria.}$$

$$y_\alpha \geq \frac{1}{a_1} + \frac{1}{a_2} \quad \Rightarrow \quad \text{The game (7) has a unique Nash Equilibrium.}$$

Thus we have proved the theorem. □

It may be worth pointing out in passing how both players behave in the case $y_\alpha \geq \frac{1}{a_1} + \frac{1}{a_2}$. If a strategy $\lambda_{2,0}$ is picked by power producer 2, power producer 1 selects the strategy $\lambda_{1,0} := \xi_1^{-1}(\lambda_{2,0})$. However,

then $\lambda_{2,1} := \xi_2^{-1}(\lambda_{1,0})$ maximizes the payoff of producer 2, so producer 2 resets its strategy to $\lambda_{2,1}$. After that, producer 1 changes its strategy to $\lambda_{1,1} := \xi_1^{-1}(\lambda_{2,1})$, which again increases the payoff. This process is repeated again and again by both players, and two sequences $\{\lambda_{1,k}\}$ and $\{\lambda_{2,k}\}$ are inductively defined. From the proof of Theorem 2, it is obvious that both sequences are strictly increasing and become arbitrarily large as k becomes large. Thus both supply functions move upwards, and the marketwide supply curve shifts to the left. Though the reality is that there is a cap on the shift, the conditions allow a situation in which it is easy to raise the spot electricity price.

Corollary 3
Let $a_1 = a_2 = a$ and $b_1 = b_2 = b$. Then the non-cooperative game (7) has a Nash equilibrium if and only if $y_\alpha < \frac{2}{a}$, in which case the Nash equilibrium is unique and

$$\lambda^* = (\lambda_1^*, \lambda_2^*) = \left(-\log\left(1-\frac{a y_\alpha}{2}\right), -\log\left(1-\frac{a y_\alpha}{2}\right)\right).$$

Furthermore, the equilibrium spot price is

$$\varphi(Y, \lambda^*) = \frac{1}{\left(1-\frac{a y_\alpha}{2}\right)^2} e^{\frac{aY}{2}+b} \tag{35}$$

4. Summary and Conclusions

We consider an asymmetric duopoly model, in which the electricity demand is stochastic, and each firm strategically chooses their supply function to maximize the α-quantile of the future profit distribution. From this model, we obtained a necessary and sufficient condition for the existence of a Nash equilibrium, and show the uniqueness of this equilibrium. Furthermore, we have presented a quasi-explicit form of the equilibrium, which is described in the proof of Theorem 2. Since this model is an extended version of a previous model by Tezuka and Ishii [18], Corollary 3 corresponds to the results from that paper. While we have derived some fundamental properties of this model, a full analysis of the equilibrium is left for future investigation.

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