A Note on the Generalization of Utility Maximizing Problem in Optimal Stopping

Aiko Kurushima* and Katsunori Ano**

Abstract

We deal with the utility maximization problem in optimal stopping problem, such as the generalization of the duration problem which is a variation of the secretary problem. In Ferguson, Hardwick, and Tamaki (1993), the problem of maximizing the duration of owning the relatively best object is solved for various settings in both no-information case and full-information case. Tamaki, Pearce, and Szajowski (1998) shows the optimal strategy for multiple stopping duration problem. In the original problem, the duration is defined as the time period when the selected relatively best object remains to be relatively best, that is just before it becomes second-best. In other words, the decision maker is interested in the object only when it is a relatively best, and when the next relatively best arrives, the former relatively best is useless. Here we extend the duration to be the time period starting from accepting the relatively best and ending at the moment when the selected object becomes relatively third-best. When the object arrives, the decision maker is also interested only in the relatively best object, but it is still useful when it becomes relatively second-best. We will show the optimal stopping rule for this problem both in no-information case and full-information case.

Keywords: Optimal Stopping, Secretary Problem, Utility Maximization

I Introduction

Among the optimal stopping problems, a lot generalizations of secretary problem have been considered. In this paper, utility maximization is dealt with as an objective. First, we consider the simple problem where the utility is the total work done by the accepted applicant. In this problem, we assume that the decision maker can observe the value (work rate) of the applicants, which is called full-information case. The optimal stopping rule does not depend on the rank of the applicants. If the no-information case of the problem, we have to think about the value of the applicants related to his/her rank. It seems to be complicated to apply the rank-based utility, the extensions of the duration problem are considered as a first step.

* Department of Economics, Sophia University
E-mail: kurushima@sophia.ac.jp

** Department of Mathematical Sciences, Shibaura Institute of Technology
E-mail: k-ano@shibaura-it.ac.jp

The objective of the original duration problem is to maximize the time period of owning the relatively best object. Many variations of the duration problem are solved originally in Ferguson, Hardwick, and Tamaki (1993), and multiple stopping problem is considered in Tamaki, Pearce, and Szajowski (1998). Here we generalize the time period to that of owning the relatively best or second-best object. We treat the problem both in no-information and full-informaion case. However, the class of the stopping rule is restricted in stopping only at the relatively best applicant.

Another genaralization is solved in Szajowski and Tamaki (2006). The problem is called shelf life problem where the objective is to maximize the time period owning the relatively best or second-best. Our second problem is a special case of the shelf life problem.

II Work Maximization Problem: Full-Information Case

Here we consider a simple utility maximization problem as a generalization of full-information secretary problem. Assume that the utility is the total work of the applicant. The work is defined as a product of value and time period.

The problem is described as follows: Fixed n applicants arrive sequentially in a random order. The decision maker (DM) has to decide whether to accept or reject the applicant after the interview. DM can observe the value of the applicant, which has uniform distribution, $U(0, 1)$. The objective of DM is to maximizing the work the accepted applicant accomplishes by the time horizon $n+1$.

Let u_i and w_i denote the value and the work of ith applicant, respectively. The total work of ith applicant is defined as the product of his/her value and time period, that is,

$$w_i(x) = u_i \times (n - i + 1). \tag{1}$$

When ith applicant has the value x, $w = x \times (n - i + 1)$. We consider the expected work of the next applicant, which is given by

$$\int_0^1 w_{i+1}(y)dy = \int_0^1 P(u_{i+1} = y)y \times (n-i)dy = \frac{n-i}{2}. \tag{2}$$

So the OLA (one-stage look-ahead) function is given by

$$u_i(x) = x(n+1-i) - \frac{n-i}{2}. \tag{3}$$

Next, we can easily show the monotone property of this problem. The OLA stopping region B is described as

$$B = \left\{(i,\ x) : x(n+1-i) - \frac{n-i}{2} \geq 0\right\}, \tag{4}$$

where (i, s) denotes the state when ith applicant whose value is x arrives. The inequality $x(n + 1 - i) \geq (n - i)/2$ is solved with respect to x, and we have

$$x \geq \frac{n-i}{2(n-i+1)}. \tag{5}$$

It satisfies $u_i(x) \geq 0 \Rightarrow u_{i+1}(y) \geq 0, y \geq x, i = 1, 2, \ldots, n - 1$. Then the OLA stopping region B is shown to be closed and the problem is monotone in the sense of Chow, Robbins, and Siegmund (1971).

It means that the OLA stopping region is optimal.

Theorem 1 *Assume that n applicants arrive in a random order. DM can observe the value of each applicant, which has uniform density between 0 and 1,* $U(0, 1)$. *The objective is to maximize the total work of the accepted applicant. Then the optimal stopping time is given by*

$$\tau = \min\left\{i : u_i \geq \frac{n-i}{2(n-i+1)}\right\},$$

where u_i *is the value of ith applicant.*

III An Extension of Duration Problem

1. Extended Duration Problem: No-information Case

In this subsection, no-information case of the duration problem is generalized. The objective is to maximize the time period of owning the relatively best and the relatively second-best. Here we consider the class of the stopping rule restricted in that of only stopping at the relatively best object.

Let X_k denote the relative rank of the kth applicant. DM accept only the relatively best applicant, so DM makes decision only when kth applicant, where $X_k = 1$, arrives. Also let T_k (S_k) represent the time when the first nest relatively best (second-best) applicant arrives thereafter, respectively.

The expected duration is described in the following two cases. First, when relatively best arrives twice after kth applicant, the duration is represented as $E(T_{T_k} - k)$. Second, when relatively best arrives after kth applicant, and relatively second-best comes next, we have $E(S_{T_k} - k)$ for the duration. Transition probabilities are given by

$$P(T_k = n+1) = \frac{k}{n},$$

$$P(T_k = j) = \frac{k}{j(j-1)},\ j = k+1,\ \ldots,\ n,$$

$$P(T_{T_k} = j \mid T_k = i) = \frac{i}{j(j-1)},\ j = i+1,\ \ldots,\ n-2,$$

and

$$P(S_{T_k} = j \mid T_k = i) = \frac{i(i-1)}{j(j-1)(j-2)},\ j = i+1,\ \ldots,\ n-2.$$

Let y_k denote the duration rate of owning the relatively best or second-best when DM accept the kth applicant who is relatively best. y_k is expressed as follows:

$$\begin{aligned} y_k &= E(T_{T_k} - k) + E(S_{T_k} - k) \qquad (6) \\ &= \sum_{i=k+1}^{n-1} \sum_{j=i+1}^{n} \frac{j-k}{n} \{P(T_i = j \mid T_k = i) + P(S_i = j \mid T_k = i)\} \\ &\quad + \sum_{i=k+1}^{n-1} \frac{n+1-k}{n} \{P(T_i = n+1 \mid T_k = i) + P(S_i = n+1 \mid T_k = i)\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{n+1-k}{n}\{P(T_k=n)+P(T_i=n+1)\}\\
&=\frac{k}{n}\sum_{i=k+1}^{n-1}\frac{1}{i-1}\sum_{j=i}^{n}\frac{1}{j}+2\cdot\frac{k}{n}\sum_{i=k+1}^{n-1}\frac{1}{i-1}-2\cdot\frac{k}{n}\frac{n-k-1}{n-1}\\
&\quad+\frac{k}{n}\left\{-\frac{k}{2}\left(\frac{1}{k}+\frac{1}{k+1}-\frac{1}{n-1}-\frac{1}{n}\right)+\frac{k^2}{2}\left(\frac{1}{k(k+1)}-\frac{1}{n(n-1)}\right)\right\}\\
&\quad+\frac{k}{n}\frac{n+1-k}{n}\frac{n-1-k}{n-1}+\frac{k(n+1-k)}{n(n-1)}\\
&=\frac{k}{n}\sum_{i=k}^{n-2}\frac{1}{i}\sum_{j=i+1}^{n}\frac{1}{j}+\frac{k}{n}\sum_{i=k}^{n-2}\frac{1}{i}+\frac{k}{n}\frac{2}{n-1}-\frac{k}{n}\frac{(n-k-1)(n+k-2)}{2n(n-1)} \qquad (7)
\end{aligned}
$$

The boundary conditions are given by

$$y_n=\frac{1}{n}$$

and

$$y_{n-1}=\frac{2}{n}.$$

Note that assuming that $\sum_{k=n-1}^{n-2}f_k\equiv 0$ for any f_k, (7) satisfies $y_{n-1}=2/n$.

Next, we consider the expected duration when DM accepts the next first relatively best applicant thereafter after rejecting kth applicant. The expected duration is given by

$$\sum_{l=k+1}^{n}\frac{k}{l(l-1)}y_l=\sum_{l=k+1}^{n-1}\frac{k}{l(l-1)}y_l+\frac{k}{n(n-1)}\cdot\frac{1}{n}$$

Then the OLA function G_k is given by

$$
\begin{aligned}
G_k&=y_k-\sum_{l=k+1}^{n}\frac{k}{l(l-1)}y_l \qquad (8)\\
&=\frac{k}{n}\sum_{i=k}^{n-2}\frac{1}{i}\sum_{j=i+1}^{n}\frac{1}{j}+\frac{k}{n}\sum_{i=k}^{n-2}\frac{1}{i}+\frac{k}{n}\frac{2}{n-1}-\frac{k}{n}\frac{(n-k-1)(n+k-2)}{2n(n-1)}\\
&\quad-\frac{k}{n}\left\{\sum_{l=k}^{n-3}\frac{1}{l}\sum_{i=l+1}^{n-2}\frac{1}{i}\sum_{j=i+1}^{n}\frac{1}{j}+\sum_{l=k}^{n-3}\frac{1}{l}\sum_{i=l}^{n-2}\frac{1}{i}+\frac{2}{n-1}\sum_{l=k}^{n-2}\frac{1}{l}\right.\\
&\quad\left.-\frac{1}{2n(n-1)}\sum_{l=k}^{n-2}\frac{(n-l-1)(n+l-2)}{2n(n-1)}+\frac{1}{n(n-1)}\right\}\\
&=\frac{k}{n}\left\{\left(1-\frac{1}{n(n-1)}\right)\sum_{i=k}^{n-2}\frac{1}{i}+\sum_{l=k}^{n-3}\sum_{i=l+1}^{n-2}\sum_{j=i+1}^{n}\frac{1}{j}+\frac{2}{n-1}-\frac{1}{n(n-1)}\right.\\
&\quad\left.+\frac{1}{2n(n-1)}\sum_{l=k}^{n-2}\frac{(n-l-2)(n+l-1)}{l}-\frac{(n-k-1)(n+k-2)}{2n(n-1)}\right\}
\end{aligned}
$$

Next, we consider the monotone property of the OLA function. We show that $G_k\geq 0\Rightarrow G_{k+1}\geq 0$. First, we rewrite the OLA function as $H_k=(n/k)G_k$. Then the statement we show also is rewritten as $H_k\geq 0\Rightarrow H_{k+1}\geq 0$.

We have

$$H_{k+1}-H_k=\frac{1}{k}\sum_{i=k+1}^{n-2}\frac{1}{i}\sum_{j=i+1}^{n}\frac{1}{j}+\frac{-3n^2+5n+3k^2+k}{2n(n-1)k} \tag{9}$$

For some large $k \leq n-3$, it follows that $H_{k+1}-H_k \leq 0$, that is H_k is decreasing in k. Setting $\varphi_k \equiv k(H_{k+1}-H_k)$,

$$\varphi_{k+1}-\varphi_k=-\frac{1}{k+1}\sum_{j=k+1}^{n}\frac{1}{j}+\frac{3k+2}{n(n-1)},$$

so $\varphi_{k+1}-\varphi_k$ is increasing in k. Next we see that H_k is increasing when k is small. Since

$$\varphi_1=\sum_{i=2}^{n-2}\frac{1}{i}\sum_{j=i+1}^{n}\frac{1}{j}-\frac{3}{2}+\frac{n+2}{n(n-1)},$$

φ_1 is iscreasing in n, and we can see that $\varphi_1 > 0$ for $n \geq 9$ by direct calculation. Then it follows that (i) for $n \geq 9$, H_k becomes concave to convex according to k, (ii) when H_k is concave, it changes increasing to decreasing, and (iii) for large $k \leq n-3$ H_k is still decreasing in k.

Then we see the first state and the last state of the OLA function. We have

$$H_1=\frac{n^2-3}{n(n-1)}\sum_{i=1}^{n-2}\frac{1}{i}-\sum_{l-1}^{n-3}\frac{1}{l}\sum_{i=l+1}^{n-2}\frac{1}{i}\sum_{j=i+1}^{n}\frac{1}{j}+\frac{7n-6}{2n(n-1)}+\frac{1}{2n}-\frac{3}{4}.$$

It also follows that H_1 is decreasing in n and by direct calculation, for $n \geq 11$, $H_1 < 0$. Futhermore,

$$H_{n-3}=\frac{2n-5}{(n-2)(n-3)}+\frac{n^2-8n+10}{n(n-1)(n-2)(n-3)}+\frac{4}{n(n-1)}$$

shows that $H_{n-3} > 0$ for $n \geq 4$.

From these statements, it is shown that $H_k \geq 0 \Rightarrow H_{k+1} \geq 0$. Finally we have the next theorem.

Theorem 2 *Assume that fixed n applicants arrive in a random order. DM can observe the rank of the applicant. The objective is to maximize the duration of owning the accepted relatively best whose rank remains within two. The optimal stopping rule is to accept kth applicant who is the first relatively best where k satisfies the equation*

$$\frac{k}{n}\left\{\left(1-\frac{1}{n(n-1)}\right)\sum_{i=k}^{n-2}\frac{1}{i}+\sum_{l=k}^{n-3}\sum_{i=l+1}^{n-2}\sum_{j=i+1}^{n}\frac{1}{j}+\frac{1}{2n(n-1)}\sum_{l=k}^{n-2}\frac{(n-l-2)(n+l-1)}{l}\right.$$
$$\left.-\frac{(n-k-1)(n+k-2)}{2n(n-1)}+\frac{2}{n-1}-\frac{1}{n(n-1)}\right\}=0.$$

2. Extended Duration Problem: Full-information Case

In this subsection, full-information case of the duration problem is generalized. The objective is to maximize the time period of owning the relatively best and the relatively second-best. Here we consider the class of the stopping rule restricted in that of only stopping at the relatively best object. The value of the applicants has the uniform distribution $U(0, 1)$. Let X_i denote the value of the ith applicant.

Let $U_n(x)$ denote the expected duration of the relatively best whose rank remains within two when the time to go is n and DM accepts the relatively best applicant whose value x is the maximum value among that of the applicants arrived so far, that is $X_n = x$. $U_n(x)$ is given by

$$\begin{aligned} U_n(x) &= \sum_{k=2}^{n-1} k(k-1)(1-x^2)^2 x^{k-2} + n(n-1)(1-x)x^n - 2 \\ &= 2(1-x)\sum_{k=1}^{n-1} kx^{k-1}. \end{aligned}$$

The expected duration when DM does not accept the relatively best applicant whose value $X_n = x$ is the maximum value among that of the applicants arrived so far and accepts the next first relatively best applicant hereafter is given by

$$\sum_{k=1}^{n-1} x^{k-1} \int_x^1 U_{n-k}(y)\,dy = 2\sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k-1} j\left(\frac{1-x^j}{j} - \frac{1-x^{j+1}}{j+1}\right).$$

Then the OLA function $G_n(x)$ is

$$G_n(x) = 2(1-x)\sum_{k=1}^{n-1} kx^{k-1} - 2\sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k-1} j\left(\frac{1-x^j}{j} - \frac{1-x^{j+1}}{j+1}\right). \tag{10}$$

Setting $i = n-k$ and deviding by x^k, the inequality $G_n(x) \geq 0$ is transformed into

$$H_i(x) \equiv 2\sum_{i=1}^{n-1} \frac{1}{x^i} \left\{(n-i)(1-x) - \sum_{j=1}^{i-1} j\left(\frac{1-x^j}{j} - \frac{1-x^{j+1}}{j+1}\right)\right\} \geq 0.$$

Since $G_n(x) = 2\sum_{i=1}^{n-1} H_i(x)/x^i \geq 0$, $H_{n-1}(x) \geq 0$. Then $H_n(x) \geq 0$. Next, $G_n(x) \geq 0$ and $H_n(x) \geq 0$ lead that

$$\begin{aligned} G_{n+1}(x) &= 2\sum_{i=1}^{n} \frac{H_i(x)}{x^i} \\ &= G_n(x) + \frac{2H_n(x)}{x^n} \\ &\geq 0. \end{aligned}$$

It follows that $G_n \geq 0 \Rightarrow G_{n+k} \geq 0$, $k = 1, 2, \ldots,$. Also it is easily shown that $G_n(x) \geq 0 \Rightarrow G_n(y) \geq 0$, $y \geq x$. Finally we got that $G_n(x) \geq 0 \Rightarrow G_{n+k}(y) \geq 0$ for $k = 1, 2, \ldots,$, $y \geq x$.

Theorem 3 *For the full-information case of the duration problem where the objective is to maximize the duration of owning the relatively best or second-best, we assume that the class of stopping rule is restricted to that of stopping only at the relatively best. Then the optimal stopping rule is to accept the applicant who has the maximum $X_n = x \geq s_n$ so far when the remaing time is n, where $s_1 = 1$ and s_n, $n \geq 2$ is the unique root of the equation*

$$\sum_{i=1}^{n-1} x^{i-1} \left\{ i(1-x) - \sum_{j=1}^{n-i-1} j\left(\frac{1-x^j}{j} - \frac{1-x^{j+1}}{j+1}\right)\right\} = 0.$$

IV Conclusion

In this paper, we consider some variations of utility maximization problem. First, total work is considered as utility. The decision made by DM is based on the value of the applicant, it is indepedent of the rank. Next, an extension of duration problem is solved both in noinformation and full-information case. Duration is defined as the time period of the rank of accepted relatively best remaining within two. Here we consider the case of accepting the relatively best. It can be more generalized, such as the duration where the rank of accepted relatively best remains within m, utility based on the relative rank, and so on. There seems to be a lot of concepts of utility, and the problems would have more realistic and interesting situations.

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